This is the required family of orthogonal trajectories.

2.4.5. Application of Orthogonal Trajectories of a Family of Curves

The Application of Orthogonal Trajectories of a Family of Curve can be determined by the following examples:

Example 45: Determine the orthogonal trajectories of the family F of curves given by F: $y^2 = cx^3$, where c is a random constant.

Solution: Given family of curves is

$$F: y^2 = cx^3 \qquad \text{....(1)}$$

On differentiate equation (1) with respect to x, we obtain

$$2yy' = 3cx^2$$

$$\Rightarrow \; y' = \frac{3cx^2}{2y} = \frac{3}{2x}\frac{cx^3}{y} = \frac{3y}{2x}$$

If at point (x, y) any curve intersects orthogonally, then (if its slope is y') must have

$$y' = \frac{dy}{dx} = -\frac{dx}{dy} = -\frac{2x}{3y}; \qquad \text{i.e., } y' = -\frac{2x}{3y}$$

On solving the above differential equation, we obtain

$$\frac{3y^2}{2} = -x^2 + c; \text{ i.e., } \frac{2x^2}{3} + y^2 = c$$

Example 46: The equation y = C/x is the family of hyperbolic curves. Determine the orthogonal trajectories for these curves.

Solution:

1) For the given family of hyperbolas determine the differential equation. Differentiating the equation with respect to x gives:

$$y' = -\frac{C}{x^2}.$$

Here we remove the parameter C from the system of two equations:

$$\begin{cases} y = \dfrac{C}{x} \\ y' = -\dfrac{C}{x^2}. \end{cases}$$

Here It is following the first equation C = xy. On putting into the second equation yields:

$$y' = -\frac{xy}{x^2} = -\frac{y}{x}.$$

2) Replace y' with $\left(-\frac{1}{y'}\right)$ we get;

$$-\frac{1}{y'} = -\frac{y}{x}, \Rightarrow y' = \frac{x}{y}.$$

3) Here we integrate the differential equation of the orthogonal trajectories:

$$y' = \frac{x}{y}, \Rightarrow \frac{dy}{dx} = \frac{x}{y}, \Rightarrow ydy = xdx, \Rightarrow \int ydy = \int xdx,$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C, \Rightarrow x^2 - y^2 = C$$